INTEGRABILITY and SEIBERG-WITTEN THEORY

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ABSTRACT

A summary of results is presented, which provide exact description of the low-energy $4d$ $N = 2$ and $N = 4$ SUSY gauge theories in terms of $1d$ integrable systems.

This conference is devoted to the impact that the ideas of Professor K.Kikkawa - especially those concerning duality and string field theory - have had in modern theoretical physics as well as to their interplay with the other concepts and methods. The subject of this contribution is related to duality through the remarkable achievement of N.Seiberg and E.Witten [1, 2], who used the duality to obtain the explicit answers for the low-energy effective actions of certain four-dimensional gauge theories. Below is a brief presentation of the results of papers [3, 4, 5], which - as a minimum - allow to represent the answers of [1] and [2] in a simple and compact form, and - as a maximum - should attract attention to the pertinent role that integrability (and thus abstract group theory) plays in the description of exact effective actions, namely, the role which becomes especially pronounced in the low-energy (Bogolubov-Whitham or topological) limit of quantum field theory. See refs.[6]-[14] for related developments. All the relevant references can be found in [4, 5].

The presentations of K.Intriligator and T.Eguchi at this conference allow us not to repeat all the discussion of $4d$ physics and the methods used in [1, 2]. We can directly proceed to our main subject.

1   The problem: from $4d$ to group theory

Renormalization group (RG) flow for the $4d$ $N = 2$ SUSY YM theory is schematically shown in Fig.1.
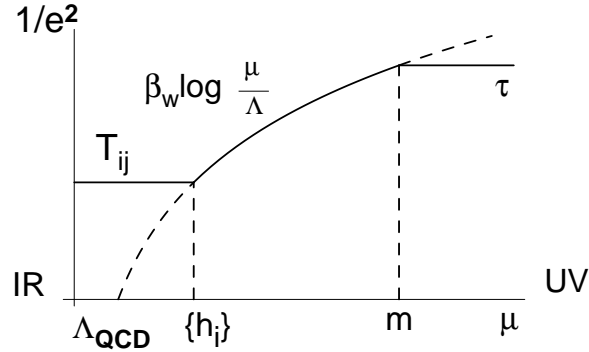
Perturbative expression $\frac{1}{e^2(\mu)} \sim \beta_W \log \frac{\mu}{\Lambda}$ receives, of course, nonperturbative corrections, which do not change the shape of the curve too much, except at the infrared (IR). We assume that theory is *softly* regularized in the ultraviolet (UV) by being embedded into some UV-finite theory, say, $N = 4$ SUSY YM. (An alternative possibility could be the $N_f = 2N_c$ model). Such regularization affects the theory for $\mu > m$. From the UV perspective m introduces the mass scale in conformally

invariant theory at $\mu \gg m$.[1]

In the IR the model is *softly* regularized by the condensates of the scalar fields, $h_k = \frac{1}{k}\langle \text{tr } \phi^k \rangle$, which break the gauge symmetry down to the pure abelian one. Then all the *minimal* couplings of fields (which are in adjoint of the gauge group) disappear. In the UV the $N = 4$ SUSY YM model is completely characterized by the gauge group G and the bare coupling constant $\tau = \frac{4\pi i}{e^2} + \frac{\vartheta}{2\pi}$ (a single complex number). The abelian model in the IR contains $r_G = \text{rank } G$ abelian gauge fields, and the corresponding complex charges form a matrix $T_{ij}, \ i,j = 1, \ldots, r_G$:

$$S^{(\text{IR})} = \int d^4x \sum_{i,j=1}^{r_G} \text{Im } T_{ij}(G^i G^j + iG^i \tilde{G}^j) + \ldots \qquad S^{(\text{UV})} = \int d^4x \ \text{Im } \tau \text{tr}(G^2 + G\tilde{G}) + \ldots$$

$$= \int d^4x d^2\theta \sum_{i,j=1}^{r_G} \text{Im } T_{ij}\mathcal{W}^i\mathcal{W}^j + \ldots \qquad = \int d^4x d^2\theta \ \text{Im}\tau \text{tr } W^2 + \ldots$$

$$= \int d^4x d^4\theta \ \text{Im } \mathcal{F}(\Phi^i) \qquad = \int d^4x d^4\theta \ \text{Im } F(\Phi) \quad .$$

$$(1.1)$$

The three lines here are written in the $N = 0$, $N = 1$ and $N = 2$ notation respectively. Only kinetic terms for the gauge (super)fields are presented explicitly and the rest is denoted by dots. The $N = 2$ (and $N = 4$) superfields in the UV are non-abelian and thus contain infinitely many auxiliary fields. At the same time the abelian superfields in the IR are equaly simple for $N = 1$ and $N = 2$ SUSY:

$$\Phi^i = A^i(\theta) + \hat{\theta}\mathcal{W}^i(\theta) + \ldots = a^i + \theta\hat{\theta}G^i + \ldots \ . \tag{1.2}$$

As a result, the abelian charge matrix is just

$$T_{ij}(a) = \frac{\partial^2 \mathcal{F}}{\partial a^i \partial a^j}. \tag{1.3}$$

Duality properties become transparent when one introduces a "dual scalar":

$$a_i^D = \frac{\partial \mathcal{F}}{\partial a^i}, \quad \text{so that} \quad T_{ij} = \frac{\partial a_i^D}{\partial a^j}. \tag{1.4}$$

It is a distinguished property of $N = 2$ gauge models that the Wilsonian background fields - of which a^i and a_i^D are examples - and not just the moduli h_i, τ, m - are physical observables. This is due to the important result of [15], which says that the mass spectrum (of BPS saturated states) - a physical observable - is exactly given by $\mathcal{M} \sim |\sum_i (m_i a^i + n^i a_i^D)|$.

The problem of defining the low-energy effective action in this framework can be formulated as follows: INPUT: G (the gauge group), τ (the UV bare coupling constant), m (mass scale)

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[1] Actually, m can be identified with the mass of the extra $N = 2$ SUSY supermultiplet in the adjoint of the gauge group, which is added to the pure $N = 2$ SUSY YM in order to convert it into the $N = 4$ SUSY one in the UV.

and h_i (symmetry breaking v.e.v.'s). OUTPUT: $a^i(h)$ (the background fields) and $\mathcal{F}(a^i)$ (the prepotential) - and thus also $a_i^D = \partial \mathcal{F}/\partial a^i$ and $T_{ij}(a) = \partial^2 \mathcal{F}/\partial a^i \partial a^j$. In other words, all what one seeks for in the setting of Seiberg-Witten theory is the RG map:

$$\{G, \ \tau; \ m, \ h_i\} \ \longrightarrow \ \{a^i(h); \ \mathcal{F}(a^i)\} \tag{1.5}$$

One can see here an analogy with the c-theorem of A.Zamolodchikov in $d = 2$ physics: the detailed description of RG flow from one fixed point (conformal model) to another is somewhat sophisticated, but the net result is simple: when we get from the UV to the IR it is enough to say that the central charge has jumped to its adjacent value (in the corresponding series of conformal models, specified by the symmetries preserved during the RG flow).

2 Intermediate data: Riemann surfaces

The discovery of N.Seiberg and E.Witten was that the mapping (1.5) is actually decomposed into two steps:

$$\{G, \ \tau; \ m, \ h_i\} \ \longrightarrow \ \left\{ \begin{array}{c} \text{Riemann surface } \mathcal{C}; \\[2mm] \text{meromorphic one} - \text{differential } dS_{\min} \text{ on } \mathcal{C} \\[2mm] \text{with the property } \dfrac{\partial dS_{\min}}{\partial h_i} = \text{holomorphic} \end{array} \right\}; \tag{2.1}$$

$$\{\mathcal{C}, \ dS_{\min}\} \ \longrightarrow \ \{a^i(h); \ \mathcal{F}(a)\}$$

The second step is simple: given $\mathcal{C}$ one can define the set of conjugate A and B cycles on it, and given $dS_{\min}$ one can write

$$a^i(h) = \oint_{A_i} dS_{\min}, \quad a_i^D(h) = \oint_{B^i} dS_{\min}, \mathcal{F}(a) = \frac{1}{2} \sum_I \oint_{A_I} dS_{\min} \int_{B^I} dS_{\min} \bigg|_{h_i = h_i(a_j)} . \tag{2.2}$$

In the last formula (the one for the prepotantial), the set of A and B contours should be enlarged to include those wrapping around and connecting the singularities of $dS_{\min}$: see [5] and section 7 below. Also, this formula, when $dS_{\min}(h)$ is substituted into , gives an answer as a function of h_i. One should further express h_i by a^i with the help of the first formula and substitute it into the last one in order to obtain the prepotential $\mathcal{F}(a^i)$.

The genus of Riemann surface $\mathcal{C}$ need not coincide with $r_G = \text{rank } G$: it is often larger. Then eqs.(2.2) seem senseless since the number of A and B contours on the r.h.s. can exceed that of the a^i and a_i^D on the l.h.s. However, in such cases - given appropriate $dS_{\min}$ (as defined in eq.(5.2)

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below) - all the extra integrals at the r.h.s. of (2.2) automatically vanish - and there are exactly r_G non-vanishing a^i - as necessary. This is of course also important to make the dependence $a^i(h)$ invertible. Thus the second mapping in (2.1) is completely described and very simple. The real issue is the first mapping in (2.1).

3   What is $\mathcal{C}$?   emergence of integrable systems

This mapping

$$\{G, \ \tau; \ m, \ h_i\} \ \longrightarrow \ \{\mathcal{C}; \ dS_{\min}\} \tag{3.1}$$

contains no reference to four dimensions, Yang-Mills theory or anything like that: it is clearly something much simpler and general. With no surprise, it can be described in a language far more primitive than that of $4d$ gauge theories: one should look for this mapping at the first place where the group theory (the input in (3.1)) meets with the algebraic geometry (the output of (3.1)). A natural place of such kind is integrability theory [16]. Namely, the map (3.1) possesses description in terms of $1d$ integrable models. In other words, the particular question (the $UV \ \longrightarrow \ IR$ map) in $4d$ gauge theory appears to be equivalent to some (actually, almost the same) question in the $1d$ integrability theory.

In this language theory (3.1) can be described as follows: Given a simple Lie group G one can construct an associated $1d$ integrable model. Parameters τ and m naturally appear in this construction. The only thing that we need on the model emerged is its Lax operator $L(z)$, which is a G^*-valued function (matrix) on the phase space of the system and depends also on the "spectral parameter" z. Thus,[2]

$$G \ \xrightarrow{\tau, m} \ L(z). \tag{3.2}$$

Then the first ingredient of the map (3.1) is

$$\mathcal{C}: \quad \det (t - L(z)) = 0. \tag{3.3}$$

If the spectral parameter z in (3.2) belongs to some complex "bare spectral surface" E, this equation defines the spectral curve $\mathcal{C}$ of $L(z)$ as a ramified covering over E. (For every $z \in E$ there are several points on $\mathcal{C}$, differing by the choice of the eigenvalue t of $L(z)$. The sheets are glued together at

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[2] The map (3.2) is actually a canonical one in the framework of geometrical quantization (Kirillov-Kostant method): it can be nicely described in terms of coadjoint orbits of G, G_q and $\hat{G}$, momentum maps, Hitchin varieties etc. What we need to know here about this map is much simpler: that it exists, is naturally defined entirely in terms of group theory, and most of explicit formulas (in convenient coordinates) are well-known since 1970's.

the points where some eigenvalues coincide.) For the given $L(z)$ eq.(3.3) depends on the integrals of motion of integrable system. These are identified with the moduli h_i at the l.h.s. of (1.5).

4 Examples

4.1 1d sine-Gordon model ($G = SL(2)$)

In this case

$$L(z) = \begin{pmatrix} p & 1 + \dfrac{1}{z}e^{-q} \\ e^q + z & -p \end{pmatrix} . \tag{4.1}$$

The Lax operator depends on the coordinate q and the momentum p, but the spectral curve (3.3) depends only on their particular combination $h_2 = p^2 + (e^q + e^{-q}) = p^2 + 2\cosh q$:

$$(t - p)(t + p) = e^q + e^{-q} + z + \frac{1}{z}, \tag{4.2}$$

or

$$z + \frac{1}{z} = t^2 - h_2. \tag{4.3}$$

This h_2 is nothing but the integral of motion (the second Hamiltonian) of the sine-Gordon system. Thus we see that what remains in (3.3) from the phase-space dependence of the Lax operator is just that on the constants of motion (this is one of the central facts in integrability theory). For us this means, that once h_i are identified with the integrals of motion of integrable system, we indeed get a map $\{G, \tau, m, h_i\} \to \mathcal{C}$.

4.2 $SL(2)$ 1d Calogero system

This time the Lax operator is expressed through elliptic functions. Elliptic functions live on elliptic *bare* spectral curve $E(\tau)$. Its modulus τ is exactly the one which is identified with the *bare* coupling constant in 4d theory. One can choose a coordinate on $E(\tau)$ in two essentially different ways: the one is the flat coordinate ξ and elliptic functions are (quasi) doubly periodic in ξ and the other is the elliptic parametrization

$$E(\tau): \quad y^2 = (x - \hat{e}_1(\tau))(x - \hat{e}_2(\tau))(x - \hat{e}_3(\tau)), \tag{4.4}$$

and $\hat{e}_1(\tau) - \hat{e}_2(\tau) = \theta_{00}^4(\tau)$, etc. The Lax operator of Calogero-Moser model is [17]

$$L(\xi) = \begin{pmatrix} p & gF(q|\xi) \\ gF(-q|\xi) & -p \end{pmatrix} \tag{4.5}$$

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where $F(q|\xi) = \frac{\sigma(q+\xi)}{\sigma(\xi)\sigma(q)}$ and Weierstrass function

$$x \sim \wp(\xi) = -\partial_\xi^2 \log \sigma(\xi) = \frac{1}{\xi^2} + {\sum_{m,n}}' \left(\frac{1}{(\xi+m+n\tau)^2} - \frac{1}{(m+n\tau)^2} \right). \tag{4.6}$$

The full spectral curve is now

$$\mathcal{C}: \quad \det(t - L(\xi)) = 0,$$

$$(t-p)(t+p) = g^2 F(q|\xi)F(-q|\xi) = g^2 \left(\wp(\xi) - \wp(q) \right), \tag{4.7}$$

or simply

$$\mathcal{C}: \quad g^2 \wp(\xi) = t^2 - h_2, \tag{4.8}$$

where this time $h_2 = p^2 + g^2 \wp(q)$ is the second Hamiltonian of Calogero-Moser system. Calogero coupling constant g is to be identified with the parameter m in $4d$ considerations:

$$g^2 \sim m^2. \tag{4.9}$$

Thus, in the framework of Calogero-Moser models we devised a mapping (3.2) parametrized by two variables, τ and $m \sim g$, which is exactly what necessary for our purposes.

5   What is $dS_{\min}$?

The last thing which is necessary to formulate our description of Seiberg-Witten theory is an explicit expression for $dS_{\min}$, which enters the formulas (2.2) for the background fields a^i and a_i^D. Now we know that the appropriate *bare* spectral surface is the elliptic curve $E(\tau)$ and the full spectral curve $\mathcal{C}$ is a ramified covering over $E(\tau)$ defined by the equation $\det(t - L(\xi)) = 0$. We are ready to give an explicit expression for $dS_{\min}$. On $E(\tau)$ there is a distinguished canonical holomorphic 1-differential

$$d\omega_0 = d\xi = \frac{1}{2\pi} \frac{dx}{y(x)} = \frac{1}{2\pi} \frac{dx}{\sqrt{x - \hat{e}_1(\tau))(x - \hat{e}_2(\tau))(x - \hat{e}_3(\tau))}} \tag{5.1}$$

Its periods on $E(\tau)$ are 1 and τ. $dS_{\min}$ is just twice the product of the Lax-operator eigenvalue t and this $d\omega_0$:[3]

$$dS_{\min} = 2t d\omega_0. \tag{5.2}$$

---

[3] Again, as in the case of (3.2), there are different interpretations of this formula: their origins range from the theory of prepotential (quasiclassical τ-functions) and Bogolubov-Whitham theory to Hamiltonian structures of integrable theories and symplectic geometry of Hitchin varieties. Again, the only thing that is essential for us is that all these interpretations are essentially the same and that the explicit formula (5.2) is true.

6 Examples

6.1 $SL(2)$ Calogero and its limiting cases

Since the curve $\mathcal{C}$ in this case is given by eq.

$$t^2 - h = g^2 \wp(\xi), \tag{6.1}$$

eq.(5.2) says that

$$dS_{\min} \sim \sqrt{h + g^2 \wp(\xi)} \, d\xi \overset{x \sim \wp(\xi)}{\sim} \frac{\sqrt{h - g^2 x}}{y(x)} dx. \tag{6.2}$$

Despite the full spectral curve $\mathcal{C}$ is of genus 2 ($\mathcal{C}$ is obtained by gluing two copies of $E(\tau)$ along one cut, which connects two points with $x = g^{-2}h$ and two different values of $y = \pm\sqrt{\prod_{a=1}^{3}(\frac{h}{g^2} - \hat{e}_a)}$). However, the differential (6.2) can be essentially considered as living on some other - genus *one* curve

$$\hat{\mathcal{C}}: \quad \hat{y}^2(x) = (h - g^2 x)(x - \hat{e}_1(\tau))(x - \hat{e}_2(\tau))(x - \hat{e}_3(\tau)), \tag{6.3}$$

which is different from $E(\tau)$ (except for the $N = 4$ SUSY limit of $h = \infty$), but of which $\mathcal{C}$ is also a double covering. In fact, this is another way of saying that $dS_{\min}$ has only two - rather than four (= twice the genus of $\mathcal{C}$) - non-vanishing periods: this is an example of how the extra periods are automatically eliminated by the choice of peculiar 1-differential (5.2). In terms of $\hat{\mathcal{C}}$ we have

$$dS_{\min} \sim \frac{h - g^2 x}{\hat{y}(x)} dx, \tag{6.4}$$

and we remind once again that $g \sim m = $ the mass of adjoint matter multiplet.

There are two interesting limits of this formula that deserve attention. The first one is the $N = 4$ SUSY limit, when the adjoint multiplet gets massless: $m^2 \sim g^2 = 0$. Then obviously

$$\hat{\mathcal{C}} \to E(\tau),$$
$$dS_{\min} \to 2\sqrt{h}d\omega_0, \tag{6.5}$$

and the periods (background fields) are

$$a = \oint_A dS_{\min} \to 2\sqrt{h}, \quad a^D = \oint_B dS_{\min} \to 2\tau\sqrt{h}. \tag{6.6}$$

Of somewhat more interest is the opposite limit, when the matter multiplet decouples, $m \to \infty$. Of physical interest is, however, the situation when the mass scale survives, i.e. the case that the dimensional transmutation takes place. This is achieved in the double scaling limit, when simultaneously

$$m^2 \sim g^2 \to \infty,$$
$$\tau \to i\infty, \quad \text{or} \quad q \equiv e^{i\pi\tau} \to 0, \tag{6.7}$$

but

$$\Lambda^{N_c} = m^{N_c} q, \quad \text{i.e.} \quad \Lambda^2 = m^2 q \tag{6.8}$$

remains finite. In this limit the *bare* spectral curve $E(\tau)$ (parametrized by ξ) degenerates into a double punctured sphere with coordinate z, and

$$\xi \to \frac{1}{2\pi i} \log \frac{z}{q}, \quad \text{and} \quad d\omega_0 = d\xi \to \frac{1}{2\pi i} \frac{dz}{z}. \tag{6.9}$$

Since $g^2 \sim \frac{1}{q}$ and $\xi + n\tau \sim \frac{1}{2\pi i} \log(zq^{2n-1})$, it is clear that the only two terms which survive in the sum

$$g^2 x \sim g^2 \wp(\xi) \sim \sum_n \frac{g^2}{\sinh^2 \pi(\xi + n\tau)} - \sum_n' \frac{g^2}{\sinh^2 \pi n\tau} \tag{6.10}$$

in the double scaling limit are those with $n = 0$ and $n = 1$, so that

$$x \sim \wp(\xi) = - \left(C(\tau) + 4q(z + z^{-1}) + o(q^2) \right). \tag{6.11}$$

Here

$$C(\tau) = \frac{1}{3\pi i} \frac{\partial \log \Delta(\tau)}{\partial \tau}, \quad \Delta(\tau) = q^2 \prod_{n=1}^{\infty} (1 - q^{2n})^{24}. \tag{6.12}$$

Accordingly, the scaling rule for the h parameter is

$$h + g^2 C(\tau) = \frac{1}{2} u, \tag{6.13}$$

and it is u that remains finite in the double sacling limit (while $h \to \infty$ as $-g^2 C(\tau = \infty) \sim -\frac{1}{3q}$). As a result,

$$dS_{\min}^{Cal} = 2\sqrt{h + g^2\wp(\xi)}d\xi \xrightarrow{d.s.l.} \frac{1}{\pi\sqrt{2}} \sqrt{\frac{\tilde{x} - u}{\tilde{x}^4 - \Lambda^4}} d\tilde{x} \sim \sqrt{u - \Lambda^2 \cos\varphi} \, d\varphi, \tag{6.14}$$

where $\tilde{x} = -\frac{1}{2}\Lambda^2(z + z^{-1}) = -\Lambda^2 \cos\varphi$. The r.h.s. of (6.14) is exactly the original Seiberg-Witten differential of [1], which describes the $N = 2$ SUSY pure gauge $SL(2)$ model. It can be of course immediately reproduced from the sine-Gordon description of our section 4.1. See [5] for more details.

6.2 Toda chain for any $SL(N_c)$ and beyond

In this case the *bare* spectral surface is a double-punctured sphere obtained by degeneration of elliptic $E(\tau)$. Other formulas from s.4.1 are generalized as follows:

$$L(z) = \vec{p}\vec{H} + \sum_{\text{simple } \vec{\alpha} > \vec{0}} \left(E_{\vec{\alpha}} + e^{\vec{\alpha}\vec{q}} E_{-\vec{\alpha}} \right) + z E_{\vec{\alpha}_0} + \frac{1}{z} e^{\vec{\alpha}_0 \vec{q}} E_{-\vec{\alpha}_0}. \tag{6.15}$$

In the fundamental representation of $GL(N_c)$ the roots are represented as matrices E_{ij} with non-vanishing entries at the crossing of i-th row and j-th column. For positive roots $i < j$ (upper

triangular matrices), for negative roots $i > j$. Diagonal matrices represent Cartan elements. The simple positive/negative roots belong to the first upper/lower subdiagonal, the affine roots $\pm\vec{\alpha}_0$ are located at the left lower/ right upper corner respectively. Thus

$$L(z) = \begin{pmatrix} p_1 & 1 & 0 & & 0 & \frac{1}{z}e^{q_1 - q_{N_c}} \\ e^{q_2 - q_1} & p_2 & 1 & & 0 & 0 \\ 0 & e^{q_3 - q_2} & p_3 & \ldots & 0 & 0 \\ & & \ldots & & & \\ 0 & 0 & 0 & & p_{N_c - 1} & 1 \\ z & 0 & 0 & & e^{q_{N_c} - q_{N_c - 1}} & p_{N_c} \end{pmatrix}.$$

The full spectral curve is given by the equation

$$0 = \det\left(t - L(z)\right) =$$

$$= (t - p_1)(t - p_2)\ldots(t - p_{N_c}) + \sum_{i=1}^{N_c} e^{q_{i+1} - q_i} + \ldots - z - \frac{1}{z} = \tag{6.16}$$

$$= \sum_{l=0}^{N_c} S_l(h_k)t^{N_c - l} = 2P_{N_c}(t|h).$$

In this formula one should take into account the periodicity condition $q_{N_c+1} = q_1$. As usual all the p, q-dependent terms gather into the Toda Hamiltonians h_k, $h_1 = \sum_i p_i$, $h_2 = \sum_i \left(\frac{1}{2}p_i^2 + e^{q_{i+1} - q_i}\right)$, $\ldots$ ($h_1 = 0$ for $G = SL(N_c)$). Finally, $S_l(h)$ are Schur polynomials. One can easily see how they appear by omitting all the interaction terms (with q-exponents) and substituting the free Hamiltonians $h_k^{(0)} = \frac{1}{k}\sum_i p_i^k$ for h_k. In order to introduce the interaction back it is enough to substitute back $h_k^{(0)} \to h_k$: all interaction effects enter only through h_k. Thus we obtain the spectral curve in the form:

$$\mathcal{C}: \quad z + \frac{1}{z} = 2P_{N_c}(t). \tag{6.17}$$

It can be brought into a more familiar form $Y^2 = P_{N_c}^2(t) - 1$ by a substitution $2Y = z - z^{-1}$. If N_f $N = 2$ SUSY matter superfields in the fundamental representation of the gauge group are added to the model in $4d$, the curve (6.17) is replaced by

$$z + \frac{1}{z} = \frac{2P_{N_c}(t)}{\sqrt{Q_{N_f}(t)}}, \tag{6.18}$$

with the polynomial $Q_{N_f}(t)$ depending on the masses of the new fields, and the same $P_{N_c}(t|h)$ as in (6.17). Finally, the 1-differential in all these cases (when the *bare* spectral curve is double-punctured sphere) is

$$dS_{\min} = \frac{t}{i\pi}\frac{dz}{z}. \tag{6.19}$$

6.3 $SL(N_c)$ **Calogero-Moser model**

This time the *bare* spectral curve is elliptic $E(\tau)$

$$
L(\xi) = \vec{p}\vec{H} + g\sum_{\vec{\alpha}} F(\vec{q}\vec{\alpha}|\xi)E_{\vec{\alpha}} =
\begin{pmatrix}
p_1 & gF_{12} & gF_{13} & & gF_{1N_c} \\
gF_{21} & p_2 & gF_{23} & \cdots & gF_{2N_c} \\
& & \cdots & & \\
gF_{N_c1} & gF_{N_c2} & gF_{N_c3} & & p_{N_c}
\end{pmatrix}.
\tag{6.20}
$$

In order to handle the elliptic functions routinely in this case one needs more relations than given in the section 4.2. Namely, a few identities for symmetric combinations of F-functions are required [4]:

$$
\mathcal{S}^2(F) = F(q|\xi)F(-q|\xi) = \wp(\xi) - \wp(q),
$$

$$
\mathcal{S}^3(F) = \frac{1}{3}\left(F(q_{12}|\xi)F(q_{23}|\xi)F(q_{31}|\xi) + \text{permutations of } q_1, q_2, q_3 \right) = -\partial_\xi\wp(\xi),
$$

$$
\cdots
\tag{6.21}
$$

$$
\mathcal{S}^r(F) = \frac{1}{r}\sum_{\text{perm}} F(q_{12}|\xi)F(q_{23}|\xi)\ldots F(q_{r1}|\xi) = (-\partial_\xi)^{r-2}\wp(\xi).
$$

With the help of these identities the full spectral curve can be represented as

$$
\mathcal{C}: \quad 0 = \det(t - L(\xi)) = \sum_{l=0}^{N_c} S_l(h)\mathcal{T}_{N_c-l}(t|\xi).
\tag{6.22}
$$

This time, in varience with the Toda-chain case, eq.(6.16), the n-th order t-polynomial $\mathcal{T}_n(t|\xi)$ - while still h-independent - depends nontrivially on ξ:

$$
t^{-n}\mathcal{T}_n(t|\xi) = 1 +
$$

$$
+ \sum_{\substack{2 \leq r_1 < r_2 < \ldots \\ m_s > 0}} \frac{n!}{(n - \sum_s m_s r_s)!}\prod_s \frac{(-)^{m_s}}{m_s!(r_s!)^{m_s}}\left(-\frac{g}{t}\right)^{\sum_s m_s r_s}\prod_s \left(\partial_\xi^{r_s-2}\wp(\xi)\right)^{m_s}
\tag{6.23}
$$

Converting this expression from the flat coordinate ξ to the elliptic ones x and $y = \sqrt{\prod_{a=1}^3 (x - \hat{e}_a)}$, one obtains

$$
\mathcal{T}_0 = 1, \quad \mathcal{T}_1 = t, \quad \mathcal{T}_2 = t^2 - x, \quad \mathcal{T}_3 = t^3 - 3xt + 2y,
$$

$$
\mathcal{T}_4 = t^4 - 6xt^2 + 8yt - 3x^2 + \sum_{a=1}^3 \hat{e}_a^2, \ \ldots
\tag{6.24}
$$

- some linear combinations of Donagi-Witten polynomials introduced in [10]. An advantage of (6.23) as compared to [10] is not only the simple derivation (and thus the possibility to obtain a general

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explicit formula (6.23)), but the full separation of h and ξ variables achieved in eq.(6.22). See [4] for details and discussion.

The 1-differential $dS_{\min}$ is, as usual, just

$$dS_{\min} = 2t(\xi)d\xi. \tag{6.25}$$

6.4 $SL(N_c)$ **Ruijsenaars Model**

To finish with our examples, we present a few formulas for the further generalization of Calogero-Moser system - its relativistic (from one point of view) or quantum group (from another perspective) generalization: the $1d$ Ruijsenaars system. The *bare* spectral curve is still elliptic $E(\tau)$, the Lax operator is composed of the already familiar elliptic functions, but it is given by a different formula [18]:

$$L_{ij}(\xi) = e^{P_i} \frac{F(q_{ij}|\xi)}{F(q_{ij}|\mu)} \prod_{l \neq i} n(\mu)\sqrt{\wp(\mu) - \wp(q_{il})}. \tag{6.26}$$

Calogero-Moser model is the $\mu \to 0$ limit of this one with the coupling constant $g \sim m$ emerging from the scaling rule for P_i-varibles. If one takes the normalization function to be $n(\mu) = \sigma(\mu) \sim \mu + o(\mu^2)$, rescales as $P_i = \frac{\mu}{g}p_i$, and takes into account that $\wp(\xi) = \mu^{-2} + o(1)$, $F(q_{ij}|\mu) = \mu^{-1}(1 - \delta_{ij}) + \delta_{ij} + o(\mu)$, it is easy to see that

$$L_{ij}(\xi) = \delta_{ij} + \frac{\mu}{g}\left(p_i + (1 - \delta_{ij})gF(q_{ij}|\xi)\right) + o(\mu^2), \tag{6.27}$$

and the order-μ term is exactly the Lax matrix (6.20). For $N_c = 2$ the full spectral curve can be written as

$$0 = \det\left(t\delta_{ij} - L_{ij}(\xi)\right) = t^2 - t \cdot \mathrm{tr}L + \det L, \tag{6.28}$$

and

$$H \equiv \frac{1}{2n(\mu)}\mathrm{tr}L = \frac{1}{2}(e^P + e^{-P})\sqrt{\wp(\mu) - \wp(q)},$$
$$\det L = n^2(\mu)\left(\wp(\mu) - \wp(\xi)\right), \tag{6.29}$$

so that (6.28) gives

$$t = \frac{H \pm \sqrt{H^2 - \wp(\mu) + \wp(\xi)}}{n(\mu)}, \quad \text{and} \quad n(\mu)dS_{\min}^{Ru} = 2n(\mu)td\omega_0 = 2H(\mu)d\omega_0 + dS_{\min}^{Cal}\big|_{h/g^2 = H^2(\mu) - \wp(\mu)}. \tag{6.30}$$

We remind that

$$dS_{\min}^{Cal} \sim \sqrt{\frac{h}{g^2} + \wp(\xi)}d\xi \sim \frac{\sqrt{\frac{h}{g^2} - x}}{y(x)}dx. \tag{6.31}$$

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7 Theory of prepotential

There are several presentations at this conference devoted to the prepotential in Seiberg-Witten theory. Instead of repeating the same things we rather outline here a general theory - not only applicable to the Seiberg-Witten case (which is associated with Riemann surfaces and corresponds to $d = 1$ and $\Omega = dS$ below). This general theory can be given different names: that of quasiclassical τ-functions, of prepotentials, the Whitham theory, special geometry etc. - see [19, 20, 21, 7, 9] for various presentations. Applications to the Seiberg-Witten case are straightforward see [5] and references therein. The real meaning of the prepotential theory - and the very fact that a more fundamental object (prepotential) than the *action* exists in a rather general setting in classical mechanics - remains obscure. It should be somehow related to the fundamental role that *quasi*periodic (rather than periodic) trajectories - which exhibit some ergodicity-like properties - play in the transition from classical to quantum mechanics. Why is the theory of quasiperiodic trajectories expressible in terms of deformations of Hodge structures - and how general this statement can be - should be a subject of further investigation: we do not touch these fundamental problems in what follows.

7.1 Notation and Definitions

Consider a family $\mathcal{M}(h)$ of complex manifolds M of complex dimension d (in the previous sections $d = 1$ and $\mathcal{M}(h)$ are some families of spectral *curves*, $M = \mathcal{C}$). The family is parametrized by some *moduli* h_k, $k = 1, \ldots, K = \dim_C \mathcal{M}$. Let us fix some canonical system of d-cycles on M: $\{A_i, B_i\}$, $i = 1, \ldots, p = \frac{1}{2} \dim H^d(M)$ with the intersection matrix $A_i \# B_j = \delta_{ij}$, $A_i \# A_j = B_i \# B_j = 0$. Finaly, pick up some holomorphic $(d, 0)$-form Ω on every M.[4] Its *periods*

$$a_i(h) \equiv \oint_{A_i} \Omega, \qquad a_i^D(h) \equiv \oint_{B_i} \Omega \tag{7.1}$$

are functions of moduli.

Consider now a variation $\delta\Omega$ of Ω with the change of parameters (moduli). $\delta\Omega$ is also a $(d, 0)$-form, not necessarily holomorphic. Still, always $\Omega \wedge \delta\Omega = 0$ (just because Ω is a maximal-rank form), and integraion of this relation over entire M gives

$$0 = \int_M \Omega \wedge \delta\Omega = \sum_i \left(\oint_{A_i} \Omega \oint_{B_i} \delta\Omega - \oint_{A_i} \delta\Omega \oint_{B_i} \Omega \right) + \text{ contribution from singularities.} \tag{7.2}$$

---

[4] It is clearly a restriction on M that such Ω exists: examples of suitable M are provided by $K3$ ($d = 2$) and Calabi-Yau ($d = 3$) manifolds. In our discussion below we shall see that this restriction can sometime be weekend, by admitting Ω's with simple singularities. Additional requirements for Ω-dependence on moduli will be specified later.

Imagine that the last item at the r.h.s. - the contribution from singularities of Ω and $\delta\Omega$ is absent. Then we obtain from (7.2):

$$\sum_i a_i \delta a_i^D = \sum a_i^D \delta a_i. \tag{7.3}$$

This implies that the *prepotential*, defined as

$$\mathcal{F} \equiv \frac{1}{2} \sum_i a_i a_i^D = \frac{1}{2} \sum_i \oint_{A_i} \Omega \oint_{B_i} \Omega, \tag{7.4}$$

possesses the following property:

$$\delta\mathcal{F} = \frac{1}{2} \sum_i \left(a_i \delta a_i^D + a_i^D \delta a_i \right) = \sum_i a_i^D \delta a_i. \tag{7.5}$$

If the freedom of variations is big enough, e.g. if $\#K = \dim_C \mathcal{M}$ is the same as $\#p = \frac{1}{2} \dim H^d(M)$, we conclude from this that

$$a_i^D = \frac{\partial \mathcal{F}}{\partial a_i} \tag{7.6}$$

and

$$\mathcal{F} = \frac{1}{2} \sum_i a_i a_i^D = \frac{1}{2} \sum_i a_i \frac{\partial \mathcal{F}}{\partial a_i}. \tag{7.7}$$

In other words, we can consider a_i as independent variables, and introduce the prepotential $\mathcal{F}(a)$ by the rule (7.4) - and it will always be a homogeneous function of degree 2 - as follows from (7.7).

The two requirements built into this simple construction are

(i) the absence of singularity contributions at the r.h.s. of (7.2);

(ii) the matching between the quantities of moduli and A-cycles,

$$K \equiv \dim_C \mathcal{M} = p \equiv \frac{1}{2} \dim H^d(M)$$

.

7.2 Comments on requirement (i)

The problem with this restriction is that variation of holomorphic object w.r.to moduli usually makes it singular - by the very definition of moduli of complex structure. Thus, even if Ω is free of singularities one should expect them to appear in $\delta\Omega$. The only way out would be to get the newly emerging poles cancelled by zeroes of Ω - but often the space of holomorphic Ω's is too small to allow for adequate adjustement.

Fortunately, requirement (i) can be made less restrictive. One can allow to consider Ω which is not holomorphic, but possesses *simple* singularities at isolated divisors. As a pay for this it is

enough to enlarge the set of A-cycles, by adding the ones wrapping around the singularity divisors, and also add all independent B-chains, connecting these divisors (such that $\partial B = div_1 - div_2$).[5] At the same time residues at the simple singularities should be added to the set of moduli $\{h\}$, thus preserving the status of the second requirement (ii). This prescription is still not complete, because the integrals over newly-added B-chains are divergent (because these end at the singularities of Ω). However, the structure of divergence is very simple: if a cut-off is introduced, the cut-off-dependent piece in $\mathcal{F}$ is exactly quadratic in the new moduli - and does not depend on the old ones. If one agrees to define the prepotential - which is generic homogeneous function of order two - modulo *quadratic* functions of moduli, the problem is resolved. Thus the real meaning of constraint (i) is that $\delta\Omega$ should not introduce new singularities as compared to Ω - so that we do not need to introduce new cycles, thus new moduli, derivatives over which would provide new singularities. Since now the freedom to choose Ω is big enough, such special adjustement is usually available.

The non-simple singularities (higher-order poles at divisors) should be resolved - i.e. considered as a limit of several simple ones when the corresponding divisors tend to coincide. The corresponding B-chains shrink to zero in the limit, but integrals of Ω over them do not vanish, if Ω is indeed singular enough. This procedure of course depends on a particular way to resolve the non-simple singularity. Essentially, if we want to allow the one of an arbitrary type on the given divisor, it is necessary to introduce coordinate system in the vicinity of the divisor and consider all the negative terms of Laurent expansion of Ω as moduli, and "weighted" integrals around the divisor as A-cycles. In the case of $d = 1$, when the divisors are just points (punctures) one can easily recognize in this picture the definition of KP/Toda-induced Whitham prepotential with one-parameter set of "time"-variables (Laurent expansion coefficients or moduli of coordinate systems) for every puncture as additional moduli (see, for example, [7, 5] and references therein). As often happens, it is most natural from the point of view of string theory (integrability theory in this case) to put all the moduli in a single point (or two), but from the point of view of algebraic geometry it is better to redistribute them as simple singularities at infinitely many divisors.

Finally, singularities of Ω on subspaces of codimension higher than one do not contribute to eq.(7.2) at all - and often variation w.r.to moduli produces only singularities of such type as $d > 1$.

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[5] For example, if M is a complex curve ($d = 1$), Ω can be a meromorphic $(1,0)$-differential with *simple* (order one) poles at some punctures ξ_α, $\alpha = 0, 1, \ldots, r$. Then one should add r circles around the points $\xi_1, \ldots, \xi_r$ to the set of A-cycles, and r lines (cuts) connecting ξ_0 with $\xi_1, \ldots, \xi_r$ to the set of B-contours in eq.(7.2). Then the last term at the r.h.s. can be omitted in exchange for enlarging the sum in the first term.

7.3 Requirement (ii)

Thus, what essentially remains is the other requirement (ii) - the matching condition between the number of moduli and A-cycles. Since the procedures involved in resolution of (i) do not change this matching (they always add as many new moduli as new A-cycles), this requirement can be analyzed at the very beginning - before even introducing Ω.

8 Picard-Fuchs equations

Dependence $a^i(h)$, $a_i^D(h)$, described by eq.(7.1) can be also formulated in terms of differential Picard-Fuchs equations for the cohomology classes of Ω. They are often convenient for comparison of Whitham universality classes of different models: Whitham-equivalent models should have equivalent Picard-Fuchs equations. We refer to [5] on details of how this idea can be elaborated on. Here we just list some important examples (all for $G = SL(2)$).

8.1 Pure gauge $N = 2$ SUSY model in $4d$: sine-Gordon model in $1d$

In this case

$$dS_{\min} \sim \sqrt{\frac{\tilde{x} - u}{\tilde{x}^2 - \Lambda^4}} d\tilde{x} \tag{8.1}$$

and Picard-Fuchs equation is [22]

$$\left(\frac{\partial^2}{\partial u^2} + \frac{1}{4(u^2 - \Lambda^4)}\right) \oint dS_{\min} = 0. \tag{8.2}$$

8.2 The flow from pure gauge $N = 4$ SUSY model to the $N = 2$ SUSY one in $4d$: Calogero-Moser model in $1d$

Now

$$dS_{\min} \sim \sqrt{\frac{\hat{h} - x}{(x - \hat{e}_1)(x - \hat{e}_2)(x - \hat{e}_3)}} dx. \tag{8.3}$$

The branching points $\hat{e}_a$ of the elliptic *bare* spectral curve $E(\tau)$ are functions of τ, and the simplest Picard-Fuchs equation looks like

$$\frac{1}{2\pi i}\frac{\partial}{\partial \tau} \oint dS_{\min} = \left(y^2(\hat{h})\frac{\partial^2}{\partial \hat{h}^2} + \left[\frac{1}{2}\hat{h}^2 - \frac{1}{2}\hat{h}C(\tau) - \frac{1}{12}g_2(\tau)\right]\frac{\partial}{\partial \hat{h}}\right) \cdot \oint dS_{\min} \tag{8.4}$$

Here $C(\tau)$ is just the same quasimodular form (the logarithmic derivative of the Dedekind function) that appeared in (6.12) above, while the modular form $g_2(\tau) = \frac{2}{3}\left[\theta_{00}^4(\tau) + \theta_{01}^4(\tau) + \theta_{10}^4(\tau)\right]$. The

differential operator eq.(8.4) is presumably convertible (by conjugation and change of variables) to the Schroedinger form:

$$\frac{1}{2\pi i}\frac{\partial}{\partial \tau} - \left(y(\hat{h})\frac{\partial}{\partial \hat{h}}\right)^2 - \hat{h} = \frac{1}{2\pi i}\frac{\partial}{\partial \tau} - \frac{\partial^2}{\partial \chi^2} - \wp(\chi),\tag{8.5}$$

which (if true) would reflect the interpretation of Whitham theory as that of quantization: (we consider essentially classical Calogero-Moser model, but the Picard-Fuchs equation on the moduli space is the Schroedinger equation for this model). See [5] for more details.

One can derive an infinite set of Picard-Fuchs equations, with different powers of τ-derivative, eq.(8.4) being the simplest one in the series (first τ-derivative). Only two of them are algebraically independent, because $\oint dS_{\min}$ depends only on two variables: $\hat{h}$ and τ. Still, the entire infinite series, once derived, can exhibit some new nice structure - as it usually happens (compare with the Virasoro etc constraints in matrix models).

8.3 Ruijsenaars model in $1d$

$$dS_{\min}^{Ru} = H(\mu)\frac{dx}{\sqrt{(x-\hat{e}_1)(x-\hat{e}_2)(x-\hat{e}_3)}} + dS_{\min}^{Cal}\big|_{\hat{h}=H(\mu)^2-\wp(\mu)}.\tag{8.6}$$

Here $dS_{\min}^{Cal}$ is given by (8.3). Picard-Fuchs equation is not drastically different from (8.4), most important, the lowest equation seems to be still of the first order in $\partial/\partial\tau$ - what does not allow to identify it with the Picard-Fuchs equation for Calabi-Yau model, where all the derivatives are of the second order (see [5] and below). Again, there are only two independent Picard-Fuchs equations.

8.4 The $WP_{1,1,2,2,6}^{12}$-induced Calabi-Yau model

The manifold is a factor of the one, defined by the equation

$$0 = p(z) = \frac{z_1^{12}}{12} + \frac{z_2^{12}}{12} + \frac{z_3^6}{6} + \frac{z_4^6}{6} + \frac{z_5^2}{2} + \phi\frac{z_1^6 z_2^6}{6} + \psi z_1 z_2 z_3 z_4 z_5\tag{8.7}$$

The 3-form Ω, which is used in the construction of the prepotential on the lines of s.7, is a restriction of $dz_1 \wedge dz_2 \wedge dz_3 \wedge dz_4 \wedge dz_5$ (one should take into account the quasihomogeneity of (8.7) - this allows to eliminate *two* variables to get a 3-form). Thus its periods are proportional to

$$\oint \Omega \sim \int \mathcal{D}\lambda(z) \int dz_1 \ldots dz_5 e^{i\lambda(z)p(z)} \sim \int dz_1 \ldots dz_5 \left(\int d\lambda e^{i\lambda p(z)}\right) \sim$$
$$\sim \int dz_1 \ldots dz_5 e^{p(z)}\tag{8.8}$$

(the quasihomogeneity of $p(z)$ is used to eliminate λ) and satisfy the set of Picard-Fuchs equations (which are nothing but Ward identities for the integral (8.8)). The simplest one is

$$\left[\left(\frac{\partial}{\partial\phi}\right)^2 - \left(\phi\frac{\partial}{\partial\phi} + \frac{1}{6}\psi\frac{\partial}{\partial\psi} + \frac{1}{6}\right)^2\right]\oint\Omega = 0. \tag{8.9}$$

Again, since there are two moduli, only two equations from the whole set will be algebraically independent.

In the particular ("conifold") double scaling limit, when $\phi, \psi \to \infty$ with $\frac{u}{\Lambda^2} = \phi - i\psi^6$ fixed, this equation (8.9) reduces exactly to the sine-Gordon-case one, eq.(8.2). This reflects the fact that in the target space this limit cortresponds to the $\alpha' \to 0$ limit, when the $d = 10$ Calabi-Yau model reduces to the $d = 4$ one - and the sector described by the periods of Ω is exactly the gauge sector described by the Seiberg-Witten theory. See [23] for details and references.

There is no doubt that the equation (8.9) itself, not only its conifold limit, can be represented in terms of some simple $1d$ system. However, at the moment we do not know what this system is. Neither Calogero nor Ruijsenaars models seems to suit. Technically, the lowest Picard-Fuchs equations for these models contain only first derivative w.r.t. one of the variables (τ), while in (8.9) both derivatives are of the second order. Physically, the relevant models should not be assocaited with particular groups (only the rank of the group should be fixed): this is because the variation of moduli of Calabi-Yau model can change one group for another (in one point of the moduli space one can have $SL(3)$ symmetry, while in another one it would be $SL(2) \times SL(2)$ - and neither one is a subgroup of another). This phenomenon is not *directly* relevant for the rank-one example of eq.(8.9) - still it explains why Calogero model itself should not be enough - and shows the direction for the search of the relevant models.

9   Instead of conclusion

Following refs.[3, 4, 5] we presented some evidence that the results like those of [1, 2] can be nicely systematized in the *a priori* different language - that of the $1d$ integrable systems. We do not find it very surprising, because the question that was addressed in [1, 2] is very special: the one about the *low-energy* effective actions, and the adequate terms in which the conformally invariant theories in the deep UV are related to the topological ones in the deep IR are necessarily rather simple.

In fact, the general scheme that one can keep in mind is as follows [24]: exact Wilsonian effective actions, defined by the functional integrals like

$$e^{S_{\text{eff}}(t|\Phi)} = \int_\Phi \mathcal{D}\phi \, e^{S_t(\phi)} \tag{9.1}$$

naturally depend on two kinds of variables: the coupling constants t_{k_x} in the bare action $S_t(\phi) \sim \sum_{\{k_x\}} t_{\{k_x\}} \mathrm{tr} \prod_x \phi_x^{k_x}$ and the background fields Φ (examples of the latter ones are our a^i and a_i^D above). Such exact effective actions (the generating functionals for all the correlators in the given field theory) - as one knows well from the example of matrix models - are "infinitely symmetric" because of the freedom to change integration variables. This symmetry is often enough to identify them with pure algebraic objects: generalized τ-functions, defined as generating functions of all the matrix elements of a universal group element g, $\tau(t|g) = \sum_{k_x, \bar{k}_x} \langle k_x | g | \bar{k}_x \rangle t_{\{k_x, \bar{k}_x\}}$. In general case (non-vanishing normalization point) both effective action and the τ-function (for quantum group) are operator-valued; the IR stable point of RG flow should then correspond to the classical limit in the group theory language.

Such considerations are, of course, very general and can seem almost senseless: still they imply something both in the general framework (for example, so defined τ-functions always satisfy some bilinear Hirota-like equations), and in concrete examples. The most famous example is the one of matrix models. Another - newly emerging example - is that of the low-energy theories: restricting consideration to the IR stable points of renormalization group flow, one drastically diminishes the number of degrees of freedom (moduli) - what in the group-theory language corresponds to consideration of small enough groups (not necessarily the 3-loop group, as at generic normalization point for the $4d$ field theory).[6]

To put it differently, various theories flow to the same universality class in the IR limit - thus these classes can be (and are) rather simple. What the general identification of effective actions with the tau-functions (i.e. with group theory) teaches us is that these classes should be also representable by some τ-functions. However, these cannot be just conventional τ-functions - defined in the Lie-group terms - because some parameter of the effective action (the normalization point) is fixed. But in order to understand *what* are these relevant objects one can consider just the RG flow within some simple enough integrable system - and then discover that the relevant objects are *quasiclassical* τ-functions (or prepotentials). This can provide a kind of an explanation of why it was natural to *try* to identify the results of [1, 2] with those of integrability theory and where exactly (the Whitham theory) one had to look for this identification. This also explains why there

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[6] A nice particular example of the relation between RG flows and integrability theory is by now famous identity

$$\beta_W \langle \mathrm{tr} \ \phi^2 \rangle \ \sim \ 2\mathcal{F}_{\mathrm{red}} - \sum_i a^i \frac{\partial \mathcal{F}_{\mathrm{red}}}{\partial a^i},$$

- a member of the anomaly family (together with $\beta_W \langle \mathrm{tr} \ G^2 \rangle \sim T_{\mu\mu}$ and axial anomaly), where the l.h.s. is clearly of RG nature and the r.h.s. represents the breakdown of homogeneity of the prepotential $\mathcal{F}$ which occurs by fixing one of its arguments (the scale Λ).

should be no big surprise once such correspondence is established.

What needs to be understood, however, is the general description of how group theory (represented by generalized τ-functions) always flows to that of Hodge deformations (represented by prepotentials). The main message of this presentation can be that such phenomenon exists, and one should think of what could be the reasons behind this and what is the adequate technical approach to a more generic situation. Once found, the answers can shed new light on the implications of symmetries (group theory) for the low-energy dynamics and algebraic geometry (of moduli spaces) - and this would be of definite use for the future developement of quantum field and string theory.

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A.Morozov, *String Theory, what is it*, Rus.Physics Uspekhi **35** (1992) 671 (v.**62**, *No.8*, p.84-175 of Russian edition);
A.Morozov, *Integrability and Matrix Models*, ibid. **164** (1994) *No.1*, 3-62 (Rus.ed.), hep-th/9303139.